Journal of Statistical Physics, Vol. 126, Nos. 4/5, February/March 2007 (© 2007)
DOI: 10.1007/s10955-006-9196-6

On the Dynamics of Kac *p*-Spin Glasses

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Received April 3, 2006; accepted August 15, 2006
Published Online: February 1, 2007

This paper discusses the dynamical properties of *p*-spin models with Kac interactions. For large but finite interaction range R one finds two different well separated time scales for relaxation. A first short time scale, roughly independent of R, on which the system remains confined to limited regions of the configuration space and an R dependent long time scale on which the system is able to escape from the confining regions. I will argue that the R independent time scales can be described through dynamical mean field theory, while non-perturbative new techniques have to be used to deal with the R dependent scales.

KEY WORDS: Spin glasses, *p*-spin, glass transition, dynamics

1. INTRODUCTION

Mean-Field spin glass model, describe glassy phenomena as sharp ergodicity breaking transitions.[1] It is well known that in that context two basic route to glassiness are found. In the SK and similar models, one finds second order phase transitions with a diverging susceptibility and a continuous order parameter.[2] In *p*-spin models, one finds a transition, of the second order kind from a thermodynamic point of view displays a first order jump in the order parameter.[3] The SK model was proposed as possible starting points to understand the physics of spin glass materials. In this contribution will concentrate instead on *p*-spin like models, which have deserved a lot of attention in the theoretical physics community as possible mean-field schematization of the structural glass transition of supercooled liquids.[4] What makes these models interesting in this connection is the presence a non-ergodic phase where the partition function is dominated by

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an exponential number of metastable states. In this phase, entered at a transition temperature T_d, the relaxation time to equilibrium is infinite and off-equilibrium dynamics falls into an asymptotic aging states with internal energy extensively higher then the equilibrium one.[5]

As the Cortona meeting has testified, progresses have been recently achieved in the comprehension of both the thermodynamics and the dynamics of Mean-Field models. On one hand, the Guerra.[6] and Talagrand.[7] analysis shows that Parisi ansatz, which in these models take the simple one step form, describes well the thermodynamics, on the other Ben Arous *et al.*[8] could prove that the physicist's equations—and underlying assumptions leading to their derivations—can be fully mathematically justified.

The situation is much less clear as one wants to deal with systems with finite range interactions. The low temperature properties of finite range spin-glass models, despite remarkable attempts of mathematical analysis.[9] remain an open issue of scientific discussion. While there is no consensus about the fate of mean-field thermodynamical spin glass phases when the range of interaction is finite, it is clear that the metastability phenomena associated to the dynamical transition in p-spin models should become cross-overs as soon as one goes to systems with finite range of interaction. Possible scenarios for this cross-over have been put at the basis of a phenomenological theory of the glass transition known under the name of 'random first order transition scenario' which describes activated processes responsible for ergodicity restorations in terms of an effective droplet model where a bulk restoring force proportional to the configurational entropy competes with interface free-energy terms that oppose to relaxation.[4,10–12] Computations of the resulting effective barrier in the context of microscopic models have been presented in Refs. 13 and 14 Many researchers believe that an accomplished microscopic theory of this cross-over could provide important hints about the relaxation processes in supercooled liquids and the glass transition.[15]

In order to understand the relation between mean field and finite range systems one can use the classical tool of Kac models,[16] where the interaction range R is considered a tunable parameter. One would like first to understand the equilibrium and dynamics in the Kac of infinite interaction range.[16,17] Then to study the dynamics and metastability effects in an asymptotic expansion in R. Recently a full mathematical control of the Kac limit for thermodynamics has allowed to prove convergence of free-energy and local correlations to the the mean-field values in p-spin models for even p.[18,19] In this paper I would like to extend the analysis to the dynamics. In particular I will consider the case of the p-spin model and study: (a) the dynamics in the Kac limit (b) the form that the relaxation time should take in an asymptotic expansion in the range of interaction. This will lead to a picture where the long time relaxation can be seen as a passage from metastable state to metastable state, and the relaxation time computed as a decay rate. The analysis I will present will rely strongly on heuristic derivations. I believe that the analysis

of the Kac limit, clear from the physical point of view, can be made fully rigorous adapting the techniques developed in Ref. 8 for the mean field case. Conversely, the dynamics on the large, R dependent time scales, remains to a large extent a challenge for theoretical physics, and the mathematical analysis seems to me far away in time.

In several occasions in this paper I will refer to the dynamical and static properties of the mean-field p-spin model as they are known from the physicist's analysis. For many of these properties mathematical proofs are lacking. Throughout my reasoning I will ignore this fact and assume that the mentioned properties are correct.

Some of the results presented in this paper have appeared in a condensed form in Refs. 13 and 20.

2. THE KAC SPHERICAL p-SPIN MODEL

In order to study the relaxation in a Langevin setting, I use the locally spherical version of the Kac p-spin model introduced in Ref. 18. Consider $\Lambda_L = \{1, 2, \ldots, L\}^d$ the hyper-cubic lattice of size L and periodic boundary conditions, partitioned into hyper-cubic boxes B_x of size ℓ (L/ℓ is a large integer), $x \in \Lambda_{L/\ell}$. The model is defined in terms of real variables $S_i \in R$ subject to the local spherical constraint $|\ell^{-d} \sum_{i \in B_x} S_i^2 - 1| < \epsilon$ for some small ϵ and interacting through the random Hamiltonian

$$H(S) = - \sum_{(i_1,\ldots,i_p) \in \Lambda_L^p} J_{i_1,\ldots,i_p} S_{i_1} \ldots S_{i_p} \tag{1}$$

where the couplings $J_{i_1,\ldots,i_p}$ are for each p-uple of the lattice i.i.d. normal variables with zero mean and variance

$$E\left(J_{i_1,\ldots,i_p}^2\right) = \frac{1}{R^{pd}} \sum_{l \in \Lambda_L} \psi(|l - i_1|/R) \ldots \psi(|l - i_p|/R). \tag{2}$$

The function $\psi : R^+ \to R^+$, is chosen to be positive $\psi(x) \geq 0$ and normalizable, $\int_0^\infty \psi(x)dx = 1$. Notice that only variables that are at distances of order R or lower can effectively interact. The usual mean-field model is recovered choosing $R = L$ and $\psi(x) = 1_{x \leq 1}$. I will consider instead the following regimes: (a) the Kac limit where $L \to \infty$, $R \to \infty$, $\ell \to \infty$ in the specified order, (b) the regime

$$L \gg R \gg \ell \gg \log(L) \tag{3}$$

where non trivial finite dimensional effects can be expected to take place.[21] Consider the Langevin dynamics

$$\frac{dS_i(t)}{dt} = -\mu_x(t) S_i(t) + p \sum_{(i_2,\ldots,i_p) \in \Lambda_L^{p-1}} J_{i,i_2,\ldots,i_p} S_{i_2}(t) \ldots S_{i_p}(t) + h_i(t) + \eta_i(t)$$

(4)

where the $\mu_x(t)$ form a set of Lagrange multipliers chosen in such a way to enforce the spherical constraint in average at each time: $E\langle \ell^{-d} \sum_{i\in B_x} S_i(t)^2\rangle = 1$, and $\eta_i(t)$ is white noise with amplitude $\sqrt{2T}$ and $h_i(t)$ is an arbitrary extra field term that will be used to generate response functions. The equation will be interpreted in the Ito sense, and will be thought to be regularized at small times (e.g. discretized) so that a regular solution-differentiable with respect to h- will exist at all times.

Below the temperature $T_d = \sqrt{(p(p-2)^{p-2}/(p-1)^{p-1})}$ the mean field model is in a non ergodic regime. One can expect for finite R the existence of two kind of time scales: time scales independent of R, which should coincide with the ones of the corresponding mean field model and are ineffective to relax at low temperature, and R dependent time scales on which there is ergodicity restoration. In order to understand the dynamics of the systems, it is interesting to consider its evolution in the Kac limit, where, in the order, $L \to \infty$, $R \to \infty$, $\ell \to \infty$ for fixed value of time. I would like to argue that for uniform initial conditions, the local correlation and response functions, defined respectively as

$$C_x^{L,R,\ell}(t,u) = \frac{1}{\ell^d} \sum_{i\in B_x} S_i(t) S_i(u)$$

$$R_x^{L,R,\ell}(t,u) = \frac{1}{2T\ell^d} \sum_{i\in B_x} S_i(t)\eta_i(u) \tag{5}$$

tend with probability one to non-random functions $C(t,u)$ $R(t,u)$ of the time arguments t and u, independent of x that verify the usual equations describing the dynamics of the mean field model (see e.g. Ref. 5 for their detailed form and a review of mean-field off-equilibrium dynamics).

Unfortunately, I cannot at the moment offer a proof of this statement. I would like however to discuss the indications that this is the case. The derivation parallels derivation in the usual mean field case, and shows that large R in the Kac case is analogous to large N in the mean field case.

2.1. An Exact Equation

In this Sec. 1 derive an exact equation, valid for all values of the parameters and all kind of initial conditions, that is a consequence of the Gaussian distribution of the couplings. Consider observables $A[S,\eta]$ depending on the values of the variables and the Brownian noise at different times. Choices of interest will be $A[S,\eta] = S_i(u)$ and $A[S,\eta] = \eta_i(u)$. I will denote with angular brackets $\langle\cdot\rangle$ the average over the initial conditions and the thermal noise, and with $E(\cdot)$ the average over the quenched couplings $J_{i_1,\ldots,i_p}$. From the Langevin equation, using partial

integration on the Gaussian couplings one gets:

$$E\left\langle \frac{dS_i(t)}{dt} A[S,\eta]\right\rangle = -E\langle \mu_x(t) S_i(t) A[S,\eta]\rangle + \langle A[S,\eta](\eta_i(t)+h_i(t))\rangle$$
$$+ \sum_{(i_2,\dots,i_p)\in\Lambda_L^{p-1}} E(J_{i,i_2,\dots,i_p})^2 E\frac{\partial}{\partial J_{i,i_2,\dots,i_p}}$$
$$\times\langle A[S,\eta] S_{i_2}(t)\dots S_{i_p}(t)\rangle \tag{6}$$

In order to make explicit the J derivative, one can use the Martin–Siggia–Rose, Girsanov representation of the joint probability of paths $S_i(t)$ and Brownian noise $\eta_i(t)$ starting from an initial condition $S_i(0)$ chosen with probability $\mu(S(0))$

$$P[S,\eta]\mu(S(0)) = \int_{-\infty}^{\infty}\prod_{u=0}^{t}\frac{d\hat{S}_i(u)}{2\pi}\exp\left(\sum_{i\in\Lambda}\int_0^t du\left\{ i\hat{S}_i(u)\left[\dot{S}_i(u)+\mu_x(u)S_i(u)\right.\right.\right.$$
$$\left.+\sum_{(i_2,\dots,i_p)} J_{i,i_2,\dots,i_p} S_{i_2}(u)\dots S_{i_p}(u) + h_i(u)+\eta_i(t)\right]$$
$$\left.\left.-\frac{1}{2T}\eta_i^2(u)\right\}\right)\mu(S(0)) \tag{7}$$

and observe, through integration by part, that insertion of $i\hat{S}_i(u)$ in correlation functions of the kind $\langle B[S,\eta]\rangle$, which improperly will be denoted as $\langle i\hat{S}_i(u)B[S,\eta]\rangle$, acts as differentiation with respect to $h_i(u)$, or, in turn, as insertion of $\frac{1}{2T}\eta_i(u)$

$$\langle i\hat{S}_i(u)B[S,\eta]\rangle = \frac{\partial}{\partial h_i(u)}\langle B[S,\eta]\rangle = \frac{1}{2T}\langle B[S,\eta]\eta_i(u)\rangle \tag{8}$$

so that,

$$E\frac{\partial}{\partial J_{i,i_2,\dots,i_p}}\langle A[S,\eta]S_{i_2}(t)\dots S_{i_p}(t)\rangle$$
$$= \int_0^t dv\; E\left(\frac{1}{2T}\langle \eta_i(v)S_{i_2}(v)\dots S_{i_p}(v)A[S,\eta]S_{i_2}(t)\dots S_{i_p}(t)\rangle\right.$$
$$\left.+\sum_{l=2}^{p}\langle S_i(v)S_{i_2}(v)\dots\eta_{i_l}(v)\dots S_{i_p}(v)A[S,\eta]S_{i_2}(t)\dots S_{i_p}(t)\rangle\right)$$
$$+E\left(\left\langle\frac{\partial\log(\mu(S(0)))}{\partial J_{i,i_2,\dots,i_p}}A[S,\eta]S_{i_2}(t)\dots S_{i_p}(t)\right\rangle\right). \tag{9}$$

2.2. The Off-Equilibrium Case

If one chooses an initial distribution independent of the quenched coupling, so that $\frac{\partial \log(\mu(S(0)))}{\partial J_{i,i_2,\ldots,i_p}} = 0$, the last term in Eq. (9) is zero. I assume in this section that this is the case, and that in addition μ is a translation invariant measure. One can consider for instance the uniform measure on the set $\mathcal{I} = \{S \in R^{L^d} : |\ell^{-1} \sum_{i \in B_x} S_i^2 - 1| < \epsilon \ \forall x\}$.

Defining the empirical correlation function and response:

$$\hat{C}_l(t,u) = \frac{1}{R^d} \sum_i \psi(|l-i|/R) S_i(t) S_i(u)$$

$$\hat{R}_l(t,u) = \frac{1}{2TR^d} \sum_i \psi(|l-i|/R) S_i(t) \eta_i(u) \tag{10}$$

one can rewrite Eq. (6) as:

$$\begin{aligned} E\langle \dot{S}_i(t) A[S,\eta]\rangle = & -E\langle \mu_x(t) S_i(t) A[S,\eta]\rangle + E\langle A[S,\eta]\eta_i(t)\rangle \\ & + \frac{1}{R^d} \sum_l \psi(|l-i|/R) p \int_0^t dv \ E \\ & \times \left(\frac{1}{2T} \langle \eta_i(v) A[S,\eta] \hat{C}_l(t,v)^{p-1}\rangle \right. \\ & \left. + (p-1)\langle S_i(v) A[S,\eta] \hat{R}_l(t,v) \hat{C}_l(t,v)^{p-2}\rangle \right) \end{aligned} \tag{11}$$

This is an exact equation valid for all values of L, R, and ℓ. From this equation one sees that if for typical realization of thermal noise and initial conditions, in the Kac limit the empirical functions $\hat{C}_l(t,u)$ and $\hat{R}_l(t,u)$ are self-averaging and tend to space homogeneous (l-independent) limiting functions $C(t,u)$ and $R(t,u)$, then the Lagrange multipliers would be x independent and one would have:

$$\begin{aligned} E\langle \dot{S}_i(t) A[S,\eta]\rangle = & -\mu(t) E\langle S_i(t) A[S,\eta]\rangle + E\langle A[S,\eta]\eta_i(t)\rangle \\ & + p \int_0^t dv \ E \left(\frac{1}{2T} \langle \eta_i(v) A[S,\eta]\rangle C(t,v)^{p-1} \right. \\ & \left. + (p-1)\langle S_i(v) A[S,\eta]\rangle R(t,v) C(t,v)^{p-2} \right). \end{aligned} \tag{12}$$

where I have used $\frac{1}{R^d} \sum_l \psi(|l-i|/R) \to 1$. Being the previous formula valid for any A, it implies, in law, the effective, single site Langevin equation:

$$\dot{S}(t) = -\mu(t) S(t) + \eta(t) + \xi(t) + (p-1) \int_0^t dv \ C(t,v)^{p-2} R(t,v) S(v) \tag{13}$$

where $\xi(t)$ is a Gaussian field with correlations $\overline{(\xi(t)\xi(u))} = pC(t,u)^{p-1}$, and subject to the consistency conditions:

$$C(t,u) = \overline{(S(t)S(u))}$$
$$R(t,u) = \frac{1}{2T}\overline{(S(t)\eta(u))}. \tag{14}$$

where the bar indicates average over the distribution of the process. This process coincides with the one found in mean field dynamics and predicts that while at high temperature the system equilibrates after a finite time, at low temperatures $T < T_d$ the system fails to equilibrate and enters an off-equilibrium aging regime where the energy remains extensively higher then the equilibrium one.

As soon as one gets away from the Kac limit and the range of interaction is finite, the relaxation time to equilibrium should be finite at least in the vicinity of T_d, and down to the temperature T_K where mean-field predict a thermodynamic phase transition to an ideal glassy state. What the analysis of the Kac limit is telling, is that for large enough R the relaxation time is essentially R independent at high temperature, while it becomes R dependent for temperatures $T \lesssim T_d$.

A proof of self-averaging and existence of the limit for the empirical correlation and response in the Kac limit can be envisaged according to the lines of Ref. 8 for the mean field case. The main formal difference with respect to that case is that the white average over sites that is responsible for the self-averaging properties of the correlation and response functions in mean-field is here substituted by the weighed average (10). The self-averaging property, with existence of the limit, should appear when the number of variables participating to the average diverges. Space homogeneity follows from the hypothesis of statistical homogeneity of the initial condition.

For the analysis that will follow it is interesting to investigate the behavior of the fluctuations for finite L, R and ℓ. of correlation and response on the scale ℓ of the boxes B_x. In particular, I would like to argue that the system is homogeneous and that the typical largest local fluctuation of the correlation on finite time is small. Equation (13) suggests indeed that for fixed times t, s, typical fluctuations of correlations and responses (5) are of order $O(\ell^{-d/2})$. Assuming the existence of a finite correlation length beyond which fluctuations become effectively independent, one can naively estimate the order of magnitude of the largest fluctuation supposing of being in presence of $O((L/\ell)^d)$ Gaussian independent events of width $O(\ell^{-d/2})$. This gives a maximum fluctuation of order $O(\sqrt{\log(L)/\ell^d})$, which the hierarchical choice (3) of the length scales guarantees to be arbitrarily small. In these condition virtually no heterogeneity on the scale ℓ can be observed for short (i.e. R-independent) time.

2.3. Equilibrium Dynamics

To study the largest time scales in the system we need to understand equilibrium dynamics. As in the previous section, let us consider the Kac limit. The main difference with the off-equilibrium case is that the distribution of the initial state

$$\mu(S_0) = \frac{1}{Z} e^{-\beta H(S_0)} \tag{15}$$

now depends on the quenched couplings. In the derivation to the analogous of Eq. (12) one should keep into account that

$$\frac{\partial \log \mu(S)}{\partial J_{i,i_2,\ldots,i_p}} = \beta(S_{i_1}, \ldots, S_{i_p} - \langle S_{i_1}, \ldots, S_{i_p} \rangle_{\text{eq}}) \tag{16}$$

this gives rise to an additional term into the exact Eq. (13) for $E\langle \dot{S}_i(t) A[S,\eta]\rangle$, that then reads:

$$\begin{aligned} E\langle \dot{S}_i(t) A[S,\eta]\rangle = & -\mu_x(t) E\langle S_i(t) A[S,\eta]\rangle + E\langle A[S,\eta](\eta_i(t) + h_i(t))\rangle \\ & + \frac{1}{R^d} \sum_l \psi(|l-i|/R) p \int_0^t dv\ E \\ & \times \left(\frac{1}{2T} \langle \eta_i(v) A[S,\eta] \hat{C}_l(t,v)^{p-1}\rangle \right. \\ & \left. + (p-1)\langle S_i(v) A[S,\eta] \hat{R}_l(t,v) \hat{C}_l(t,v)^{p-2}\rangle \right) \\ & + \frac{1}{R^d} \sum_l \psi(|l-i|/R) E\langle S_i(0) A[S,\eta] \hat{C}_l(t,0)^{p-1}\rangle \\ & - \sum_{l,i_2,\ldots,i_p} \frac{1}{R^{pd}} \sum_l \psi(|l-i|/R) \prod_{r=2}^{p} \psi(|l-i_r|/R) \\ & \times \beta E \left(\langle A[S,\eta] S_{i_2}(t) \ldots S_{i_p}(t)\rangle \langle S_i S_{i_2} \ldots S_{i_p}\rangle_{\text{eq}} \right) \end{aligned} \tag{17}$$

In order to write closed equations in this case, besides hypothesizing the self-averaging condition for $\hat{C}_l$ and $\hat{R}_l$ discussed in the previous section, one needs to know the structure of the equilibrium correlations that appear in (17). The results of Ref. 18 imply that the static correlations on length scales of order R, tend to the corresponding mean field function in the limit. I discuss here to the regime $T_k < T < T_d$ where these static correlations are vanishingly small in the limit, so that the last term in (17) will be zero. In equilibrium conditions the fluctuation dissipation theorem and time translation invariance holds and $R(t,s) = R(t-s) = \beta \frac{\partial C(t-s)}{\partial s}$. The resulting equation for C reduces after a little algebra to the usual equilibrium

mean field equation:[22]

$$\frac{dC(t)}{dt} = -TC(t) + \frac{p}{2T}\int_0^t ds\; C^{p-1}(t-s)\frac{dC(s)}{ds} \tag{18}$$

The behavior of this equation is very well known. Above T_d the correlation function decays to its equilibrium value $\lim_{t\to\infty} C(t) = 0$, while below T_d, ergodicity is broken and $\lim_{t\to\infty} C(t) = q_{\mathrm{EA}}$. q_{EA} is the Edwards–Anderson parameter, which does not coincide with the static value of the correlations in the Kac limit $\lim_{R\to\infty} \frac{1}{\ell^d}\sum_{i\in B_x} E\langle S_i\rangle^2_{\mathrm{equil}} = 0$.

In this temperature region the Kac limit does not commute with the large time limit. A system with a finite but large R, for a large time will follow closely the evolution dictated by (18), but, eventually, will be able to relax below q_{EA} on R dependent time scales. Increasing R, one can make the relaxation time scale as long as wanted, in particular, it can be chosen so long that the system has the time to explore ergodically the region where the local correlations $C_x(t)$ never go below a certain value $q_0 < q_{\mathrm{EA}}$. This is a crucial observation that allows to identify typical equilibrium configurations as belonging to metastable states as it will be discussed in the next section.

3. *R* DEPENDENT TIME SCALES

On time scales diverging with the interaction range, the independence between different spins, which is implied by the self-averaging property of the correlation function, should fail and the mean-field dynamical equations loose their validity. In order to investigate the relaxation below q_{EA} I proceed in a different manner, reminiscent of the Lebowitz and Penrose[21] analysis of metastability in Kac models of first order phase transition.

As I previously discussed, as a consequence of the fact that the relaxation time is divergent with R, for R large enough the equilibrium dynamics has the time to explore ergodically the configurations such that for all the boxes B_x, the correlation verifies $C_x(t) > q_0$. This condition defines this set of configurations as metastable state in the sense of Ref. 21. In fact one has the following properties: 1) The system finds itself in states of constrained equilibrium. 2) The time to leave these states is large, but once departed much larger is the time to come back. 3) The correlation function is homogeneous in space. Formally, for typical equilibrium configurations S_i^0, one can define a metastable state the restricted equilibrium measure on the set $\mathcal{R}(S_0) = \{S \in \mathcal{I} |\; q_x(S, S^0) > q^0\}$, namely

$$\mu_{\mathcal{R}(S_0)}(S) = \frac{1}{Z_{\mathcal{R}(S_0)}} e^{-\beta H(S)} 1_{\mathcal{R}(S_0)}(S). \tag{19}$$

Notice that the states defined in this way do not form a partition of the equilibrium manifold. In fact couples of states so defined, corresponding to two different

reference equilibrium configurations can overlap if these configurations are close enough to each other. Despite this fact, one can profitably describe the relaxation as a passage from metastable state to metastable state, and estimate the relaxation time as the inverse decay rate of these states. Denoting as $\mathcal{H}$ the Fokker–Plack operator and as $\overline{\mathcal{R}}(S_0)$ the set complementary to $\mathcal{R}(S_0)$, the decay rate of $\mathcal{R}(S_0)$ is:

$$\lambda = \int_{S\in\mathcal{R}(S_0);\quad S'\in\overline{\mathcal{R}}(S_0)} dS\, dS'\ \langle S'|\mathcal{H}|S\rangle \times \mu_{\mathcal{R}(S_0)}(S). \tag{20}$$

This expression can be bounded from above noticing that the Fokker–Planck operator is local, so that if S' has to be in $\overline{\mathcal{R}}(S_0)$ then S has to be on the border of $\mathcal{R}(S_0)$,

$$\partial\mathcal{R}(S_0) = \{S \in \mathcal{R}(S_0)|\ \ \exists x_0 :\ q_{x_0}(S, S^0) = q^0\}. \tag{21}$$

Moreover, for any S, the integral $\int dS'\ \langle S'|\mathcal{H}|S\rangle$ can be bound by a constant C of order one, and one estimates

$$\lambda \le C \int_{S\in\partial\mathcal{R}(S_0)} dS\ \mu_{\mathcal{R}(S_0)}(S) = C\frac{Z_{\partial\mathcal{R}(S_0)}}{Z_{\mathcal{R}(S_0)}} \tag{22}$$

Two comments are here in order: 1) Defined in this way, the escape rate λ, and its estimate $\frac{Z_{\partial\mathcal{R}(S_0)}}{Z_{\mathcal{R}(S_0)}}$ are functions of the reference configuration S_0. One can expect however that both quantities are self-averaging, i.e. assume values independent on S_0 with probability approaching 1 in the thermodynamic limit. This suggests to consider

$$\lambda_{typ} \sim \exp\left[E(\log Z_{\partial\mathcal{R}(S_0)} - \log Z_{\mathcal{R}(S_0)})\right] = \exp(-\beta\Delta F(q_0)) \tag{23}$$

where E stands here for Boltzmann–Gibbs average over the S_0 and quenched average over the Gaussian couplings. 2) The definition of $\mathcal{R}(S_0)$ and of the decay rate depend on the value q_0, which defines a sort of amplitude of the state. The "right" value optimizing the estimate, is the one such that a local equilibrium fluctuation of size q_0 has equal probability to be reabsorbed in $\mathcal{R}(S_0)$ and to drive the system out of it. This, together with the fact that what we get is a lower bound for the decay rate, suggests that analogously to first order transition kinetics,[23] one should maximize $\Delta F(q_0)$ with respect to q_0.

We get a relation between the relaxation time and a free-energy barrier defined coupling the system with a reference configuration. On general ground one can expect the barrier to be proportional to the interaction volume R^d. In the next section I will discuss some theoretical techniques to compute the barrier.

4. THE COMPUTATION OF THE BARRIER

I would like to discuss here possible strategies for the computation of the barrier. This is a hard task and my discussion will be highly conjectural. One

needs to compute, for $\mathcal{S} = \mathcal{R}$ or $\mathcal{S} = \partial\mathcal{R}$, the following free-energy:

$$F_{\mathcal{S}} = -T \left\langle\!\!\left\langle \frac{-1}{Z} \int dS_0 e^{-\beta H(S_0)} \log \int dS e^{-\beta H(S)} 1_{\mathcal{R}(S_0)}(S) \right\rangle\!\!\right\rangle \tag{24}$$

where I denoted with $\langle\langle\cdot\rangle\rangle$ the average over the quenched couplings $J_{i_1,\ldots,i_p}$. This free-energy will consist in an extensive term plus corrections. In considering the difference, the extensive term and the corrections due to finite size effects should compensate in the difference and only the term, finite in the thermodynamic limit, which is related to the relaxation time survives. From the previous analysis, it is clear that $F_{\mathcal{R}}$ is dominated by the configurations such that in all the boxes B_x one has that $q_x(S, S_0) = q_{\mathrm{EA}}$. On the other hand, the restricted partition function $Z_{\partial\mathcal{R}(S_0)}$ can be written as a sum over all sites x_0 of partition sums restricted to the configurations such that $q_{x_0}(S, S_0) = q_0$, namely,

$$Z_{\partial\mathcal{R}(S_0)} = \sum_{x_0} Z_{x_0} \tag{25}$$

$$Z_{x_0} = \int_{S \in \mathcal{R}(S_0);\, q_{x_0}(S,S_0)=q_0} dS e^{-\beta H(S)} \tag{26}$$

One can expect Z_{x_0} to be dominated by a single configuration consisting in a localized excitation of linear size $O(R)$ around the point x_0. For large R, for most sites x_0 these excitations will become self-averaging and independent of the site x_0. The partition function $Z_{\partial\mathcal{R}(S_0)}$ will receive contributions from these sites, but could also receive contributions from exceptional sites particularly keen to excitation and where the relaxation would be initiated with the highest probability. At present I do not have a clear cut argument to decide if the dominant contribution to the barrier is given by typical or exceptional sites. In favor of the first hypothesis one could argue that the condition $\ell \gg \log L$ should guarantee that too exceptional sites are absent from typical samples. If this is the case, any site x_0 is equally likely to initiate the relaxation independently of S_0. In favor of the second one, one could argue that relaxation below q_{EA} proceeds through amplification of small $O(\ell^{-d/2})$ fluctuations in the initial condition. This kind of amplification effect has been observed in realistic short range glass models, where sites that exhibit faster then average relaxation in the fast regime, are more likely to initiate the slow part of the relaxation.[24] In that cases however the interaction range cannot be considered a large parameter.

At any rate, the free-energy $F_{\partial\mathcal{R}(S_0)}$ can be estimated through the the replica method, taking advantage of the identity

$$E(F_{\partial\mathcal{R}(S_0)}) = \lim_{\substack{m\to 0 \\ n\to 0}} -T \left\langle\!\!\left\langle Z^{m-1} \int dS_0 e^{-\beta H(S_0)} \left(Z^n_{\partial\mathcal{R}(S_0)(S_0)} - 1\right)/n \right\rangle\!\!\right\rangle. \tag{27}$$

As usual, the computation is performed starting from integer values of m and n, and continuing to real values using a an appropriate modification of Parisi ansatz to take into account the constraints. It should be possible as in case of the computation of the unconstrained free-energy, to relate the replica computations to a variational principle in the space of "random overlap structures,"[25] but in this paper I will not pursue this path. Instead, I will sketch the outline of the replica computation of the free-energy in the simplest possible approximation.[13,20,26] For integer n and m the average over the quenched noise can be explicitly performed. One has to introduce local overlaps between replicas

$$Q_x^{a,b} = \frac{1}{\ell^d} \sum_{i \in B_x} S_i^a S_i^b \quad a, b = 1, \ldots, n+m \tag{28}$$

where the replicas from 1 to $m-1$ describe the normalization Z^{m-1}, replica number m describes S_0 and replicas from $m+1$ to $m+n$ describe configurations in $\mathcal{R}(S_0)$. In terms of these order parameters one can write the replicated partition function as

$$Z_{\text{repl}} = \int \prod_{a>m} dx_0^a \int_{Q_x^{m,a} \geq q_0;\ Q_{x_0}^{m,a} = q_0 \ \forall a > m} \mathcal{D}Q e^{-\frac{1}{T} F[Q]} \tag{29}$$

where $F[Q]$ is a coarse grained replica free-energy.[19]

$$F[Q] = R^d S[Q] = R^d \int_0^{L/\ell} d^d x \ \left[K(\{Q_{\alpha,\beta}\}, x) + V(Q(x)) \right] \tag{30}$$

with

$$K(Q_{\alpha,\beta}, x) = \frac{-\beta}{2} \sum_{\alpha,\beta} [f(\hat{Q}_{\alpha,\beta}(x)) - f(Q_{\alpha,\beta}(x))]$$

$$V(Q) = -\frac{\beta}{2} \sum_{\alpha,\beta} f(Q_{\alpha,\beta}) - \frac{1}{2\beta} \text{Tr} \log Q \tag{31}$$

and I defined

$$\hat{Q}_{\alpha,\beta}(x) = \int dy \ \psi(x-y) Q_{\alpha,\beta}(y). \tag{32}$$

Notice that thanks to the scaling $F[Q] = R^d S[Q]$ for large R one can evaluate the integral (29) by saddle point. I would like here just briefly discuss the results of the simplest ansatz for the matrix $Q_{ab}(x)$, obtained assuming replica symmetry. It is clear from (27) that the integer n and m partition function is invariant under the group of $S^{m-1} \times S^n$ of permutation of the $m-1$ replicas ($a = 1, \ldots, m-1$ in our notation) coming from the denominator among themselves and the n ($a = m+1, \ldots, n+m$) replicas in $\partial\mathcal{R}(S_0)$ among themselves. Within the replica symmetric framework one chooses $x_0^a = x_0$ for all $a > m$. This ansatz for x_0^a is the translation

into the replica formalism of the hypothesis discussed above that independently of x_0 all sites are equally likely to start the relaxation. Replica symmetry also implies that following structure of the matrix $Q_{ab}(x)$:

$$Q_{a,b}(x) = \begin{cases} 1 & a = b \\ s(x) & a \neq b \;\; a, b = 1, \ldots, m-1 \\ p(x) & a < m, \;\; b > m \;\; \text{or} \;\; a > m, \;\; b < m \\ q(x) & a = m, \;\; b > m \;\; \text{or} \;\; a > m, \;\; b = m \\ r(x) & a \neq b \;\; a, b \geq m \end{cases} \tag{33}$$

It can be seen[26] that the equation for $s(x)$ decouples from the other variables and is solved by $s(x) = p(x) = 0$. As far as the other parameter are concerned, a further simplification consists in considering saddle points with $r(x) = q(x)$, which can be verified to exist.[13] With this ansatz, the various terms of the free-energy become, in the limit of small n:

$$\begin{aligned} K(\{q(x)\}, x) &= -n\frac{\beta}{2}[\hat{q}^p(x) - q^p(x)] \\ V(q(x)) &= -n\left(\frac{\beta}{2}q^p(x) + \frac{1}{2\beta}[\log(1 - q(x)) + q(x)]\right). \end{aligned} \tag{34}$$

and the action functional

$$S = \int dx[K(x) + V(x)] \tag{35}$$

$V(q)$ is a single minimum function at high temperature ($T > T_d$), while it has two minima below. The free-energy difference between the minima is known to be equal to the configurational entropy per spin $\Sigma(T)$ in the mean-field model, i.e. the logarithm of the multiplicity of ergodic components divided by the volume.[26] Within the simple approximation in which the replica matrix is parametrized in term of a single space dependent parameter, one is in presence of a Landau like field theory analogous to the one for systems with a first order transition. The free-energy barrier can the be estimated through instantonic techniques, looking at the minimum action solution with boundary condition equal to q_{EA} at infinity and to q_0 in x_0.[11,13] Close to T_k, where the two minima become degenerate, this results in a conventional droplet theory where the free-energy barrier results from an effective droplet model where a bulk free-energy, in this case proportional to the configurational entropy, competes with a boundary term, which for the nature of the theory is just a surface term. The detailed computation of the barrier.[13] leads to a Vogel–Fulcher like expression

$$\Delta F = R^d \frac{C(T)}{\Sigma(T)^{\gamma_d}} \tag{36}$$

with $\gamma_d = d - 1$ and $C(T)$ a regular function of T. This computation supports in the context of a microscopic model the phenomenological "entropic droplet" arguments first introduced in Ref. 4 and recently revived in Ref. 12. Formula (36) is obtained in the replica symmetric approximation and could be changed by considering better saddle points. About this result I have mixed feelings: On the one hand one finds a Adam–Gibbs like relation of inverse proportionality between configurational entropy and free-energy barrier, which is a good thing. On the other, in the relation one finds an exponent $\gamma_d = d - 1$ which would predict a lower critical dimension of 1 for the ideal glass transition and would be equal to 2 in three D. This is at variance with the expectation, based on the Adam–Gibbs theory and verified in numerical simulations of model glasses, of a linear relation between the free-energy barrier and inverse configurational entropy in three dimensions. I believe that while the general form $\Delta F = R^d \frac{C(T)}{\Sigma(T)^{\gamma_d}}$ only depends on the fact that the bulk free-energy difference between the minima is the configurational entropy, should be robust to improvements of the calculation, the value of γ_d could be modified by better approximations. One can envisage two sort of modifications in the replica computation: 1) Within the ansatz that x_0^a is independent of a, to find better saddle points for the matrix $Q_{a,b}(x)$. This has been attempted in Ref. 14 where an replica broken solution to the instantonic problem has been found. Unfortunately, this solution though changing the function $C(T)$, it does not change the exponent γ_d. It is my opinion that saddle points in this class cannot modify the exponent. 2) One can consider more complex configurations where the position x_0^a of the center of the instanton depends on a. Saddle points of this kind would break the replica symmetry in a vectorial way,[27] and could lead to a different value of the exponent γ_d. The search of new saddle point to the replica variational problem will be matter of future research.

5. CONCLUSIONS

In this paper I presented the beginning of the analysis of the dynamics of the spherical p-spin model with Kac kind of interactions. I argued in favor of the existence, both in off-equilibrium and equilibrium dynamics at low temperature, of two different time regimes for large R. There is a first regime where dynamical time scales are independent (or quasi-independent) of R. Evidence for this regime is reached observing that in local correlations and responses should verify the dynamical mean-field equations of the infinite range model in the Kac limit. I am confident that the dynamical equations in this regime will be fully mathematically justified. Then there is a second dynamical regime, on time scales diverging with R, where the system manages to explore ergodically the configuration space. The extreme separation of scales that one has for large R allows the description of the dynamics as a passage between metastable states that are explored ergodically before being abandoned. Thanks to that property, the relaxation time can be

estimated as the escape time from typical states and related to a well-defined free-energy barrier. The computation of this barrier is unfortunately a highly non-trivial problem. We presented here a simple attempt based on replica symmetry predicting an ideal glass transition for the mode in dimension $d > 1$ and giving a modified Adam–Gibbs relation between free-energy barrier and configurational entropy. The simple theory we have sketched could fail due to several kinds of replica symmetry breaking effects. While we believe that the general relation will continue to be valid, the value of the exponent in modified Adam–Gibbs relation and then the lower critical dimension for the ideal glass transition could be changed in the ultimate theory.

ACKNOWLEDGMENTS

This work was supported in part by the European Community's Human Potential programme under contract "HPRN-CT-2002-00319 STIPCO."

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